

Physical Significance of the Difference between the Brans-Dicke Theory and General Relativity

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Abstract

The asymptotic behavior of the scalar field and its physical meaning are discussed for $T = 0$ and $T \neq 0$ for the large enough coupling parameter ω . The special character of the Brans-Dicke theory is also discussed for local and cosmological problems in comparison with general relativity and the selection rules are introduced respectively. The scalar field by locally-distributed matter should exhibit the asymptotic behavior $\phi = \langle \phi \rangle + O(1/\omega)$ because of the presence of cosmological matter. The scalar field of a proper cosmological solution should have the asymptotic form $\phi = O(\rho/\omega)$ and should converge to zero in the continuous limit $\rho/\omega \rightarrow 0$.

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I Introduction

It seems that Einstein's general relativity has increasingly obtained its numerical validity by many experimental and observational tests. Nevertheless, on the other hand, much efforts for scalar-tensor theories of gravitation also continue for a long time. We have some historical or fundamental bases on which we believe that there should exist some kinds of scalar fields as the gravitational field.

The Brans-Dicke theory [1] is the prototype of such scalar-tensor theories of gravitation, and the gravitational field is described by the metric tensor $g_{\mu\nu}$ of the Riemannian manifold and the non-minimally coupled scalar field ϕ on that manifold, which represents the spacetime-varying gravitational "constant". The field equations of the Brans-Dicke theory are obtained by the similar variational method as the Einstein theory, and are given as following in our sign conventions:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi}{c^4\phi}T_{\mu\nu} - \frac{\omega}{\phi^2}\left(\phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\phi_{,\lambda}\phi^{,\lambda}\right) - \frac{1}{\phi}(\phi_{,\mu;\nu} - g_{\mu\nu}\square\phi), \quad (1)$$

$$\square\phi = -\frac{8\pi}{(3+2\omega)c^4}T, \quad (2)$$

where $T_{\mu\nu}$ is the energy-momentum tensor of matter and ω is the coupling parameter of the scalar field.

As mentioned in many literatures (see, for example, [2]), when the coupling parameter ω is large enough, the scalar field and the field equations of gravitation have the following approximate form:

$$\phi = \langle\phi\rangle + O(1/\omega), \quad (3)$$

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi}{c^4\phi}T_{\mu\nu} + O(1/\omega), \quad (4)$$

and in the limit of infinity ($\omega \rightarrow \infty$), the scalar field ϕ converges to constant $\langle\phi\rangle$, thus the field equations of gravitation coincide completely with those of general relativity by replacing ϕ with Newton's gravitational constant $G \equiv \langle\phi\rangle^{-1}$.

Recently, however, some authors [3], [4] reported that these discussions are generally not right when the contracted energy-momentum tensor $T = T^\mu_\mu$ vanishes. According to Banerjee and Sen [3], in this situation $T = 0$, asymptotic behavior of the scalar field becomes

$$\phi = \langle\phi\rangle + O(1/\sqrt{\omega}) \quad (5)$$

when the coupling parameter is large enough. In the limit of infinity, though the scalar field definitely converges to constant, the second term of the right-hand side of Eq.(1) remains nonvanishing and the field equations of the Brans-Dicke theory do not coincide with those of the Einstein theory with the same energy-momentum tensor $T_{\mu\nu}$. As for such examples of exact solutions, see Refs of [3], [4]. They say that *the condition $T \neq 0$ is both necessary and sufficient for the Brans-Dicke solutions to yield the corresponding solutions of general relativity with the same $T_{\mu\nu}$ in the infinite ω limit.*

However, this theorem is not true as indicated by Faraoni [4] with a counterexample [5]. Faraoni gave a rigorous mathematical proof to the asymptotic behavior Eq.(5) by discussing the conformal invariance of the Brans-Dicke theory when $T = 0$. He insists only that *the Brans-Dicke solutions with $T = 0$ generically fail to reduce to the corresponding solutions of general relativity when $\omega \rightarrow \infty$.*

In this article we will add another counterexample to the above theorem [3], and will discuss generally the physical meaning of the relationship between the Brans-Dicke theory and the Einstein theory in the cases $T = 0$ and $T \neq 0$ in contexts of local or cosmological problems. Rather, going back to the original motivation of the Brans-Dicke theory, the Machian point of view, we realize that both are different theories of gravitation, and that the Brans-Dicke theory need hardly reduce to the Einstein theory when $\omega \rightarrow \infty$ except local problems.

II Physical Meaning of the Asymptotic Behavior

Let us discuss the asymptotic behavior of the scalar field when $T = 0$. An order of magnitude estimate by Banerjee and Sen [3] is more appropriate to understand its physical meaning. When $T = 0$, we obtain from Eq.(2)

$$\square\phi = 0, \quad (6)$$

and get from the trace of Eq.(1)

$$R = -\frac{8\pi}{c^4\phi}T - \frac{\omega}{\phi^2}\phi_{,\lambda}\phi^{,\lambda} - \frac{3}{\phi}\square\phi. \quad (7)$$

It is easy to see the asymptotic behavior Eq.(5) of the scalar field from this equation when $T = 0$. However, remember we assume tacitly that the scalar field ϕ converges to constant in the infinite ω limit and the scalar curvature R does not depend on ω , both of which do not seem to be obvious. Moreover,

the Minkowski space with $T = 0$ and $R = 0$ has only the constant scalar $\langle\phi\rangle$, which is independent of the coupling parameter ω . It is to be remarked that a solution satisfying

$$\frac{\omega}{\phi^2} \left(\phi_{,\mu} \phi_{,\nu} - \frac{1}{2} g_{\mu\nu} \phi_{,\lambda} \phi^{,\lambda} \right) = 0 \quad (8)$$

is only $\phi = \text{const}$ [4]. Therefore, all Einstein spaces ($R_{\mu\nu} = 0$) with constant ϕ are exceptions for the statement by Banerjee and Sen, or by Faraoni.

When $T \neq 0$ the asymptotic form of the field equation (2) becomes

$$\square\phi = O(1/\omega), \quad (9)$$

and we observe the well-known asymptotic behavior Eq.(3) of the scalar field. We, however, should strictly read Eq.(9) as $\square\phi = O(T/\omega)$, or for simplicity

$$\square\phi = O(\rho/\omega) \quad (10)$$

for dust matter.

Now we can understand the physical meaning of the difference of the asymptotic behavior of the scalar field. The Brans-Dicke theory includes originally the coupling parameter ω in the right-hand side of Eq.(2) and in the second term of the right-hand side of Eq.(1). When $T \neq 0$, the dependence of ω in the scalar field comes fundamentally from the coupling parameter ω in the right-hand side of Eq.(2), and the second term of the right-hand side of Eq.(1) vanishes in the infinite ω limit. When $T = 0$, the right-hand side of Eq.(2) vanishes and the dependence of ω comes fundamentally from ω in the second term of the right-hand side of Eq.(1). The Brans-Dicke scalar field has finite indefiniteness $\phi_V(x^\mu)$ which satisfies the d'Alembertian equation (6) even when matter does not exist ($T = 0$). This scalar field $\phi_V(x^\mu)$ is constrained by another field equation (1), and thus it has the dependence of ω like Eq.(5). The scalar field $\phi_V(x^\mu)$, which behaves like a source of the gravitational field $g_{\mu\nu}$ in Eq.(1), has no material origin. The constant part $\langle\phi\rangle$ itself is a special case of this scalar field without material origin. To the contrary, the asymptotic behavior of the scalar field with material origin is determined by its field equation (2) with the source term.

In general, when matter exists ($T \neq 0$), the scalar field includes a part given by matter and indefiniteness $\phi_V(x^\mu)$ for $T = 0$, and its asymptotic behavior for large ω becomes

$$\phi = \langle\phi\rangle + O(1/\sqrt{\omega}) + O(1/\omega). \quad (11)$$

It is clear that the term of $O(1/\sqrt{\omega})$ is dominant when the coupling parameter ω is large enough. Therefore the Brans-Dicke theory fails to yield

general relativity in the infinite ω limit. This situation produces possible counterexamples to the theorem proposed by Banerjee and Sen. An example cited by them to reinforce the theorem, a closed vacuum ($T = 0$) Friedmann-Robertson-Walker solution with cosmological constant Λ [6], [7], should rather be included here because of $\square\phi = 2\Lambda\phi/(2\omega + 3) \neq 0$ though $T = 0$.

III Local Problems

When we consider the difference between the Brans-Dicke theory and the Einstein theory, we should realize the difference between local problems (with locally-distributed matter) and cosmological (or global) problems. Brans and Dicke [1] also comment, in discussing a Schwarzschild solution in their theory, that we premise the existence of distant matter in the universe. Our universe always exists, and in the Brans-Dicke theory we need discuss local problems in the presence of cosmological matter, which supports the gravitational "constant". In the framework of general relativity, which has *a priori* gravitational constant, we do not consider our environment of the universe and discuss purely the local gravitational field with an asymptotically-flat boundary condition.

It might be very difficult to solve globally all configurations of matter in the universe in the Brans-Dicke theory. However, it seems to be a good enough approximation to divide the two side, local and cosmological problems, because our universe is huge enough. We can discuss individual problems of locally-distributed matter with an asymptotically-flat boundary condition. To do so, we need accept two postulates; We use an experimental value of gravitational constant for the constant scalar field $\langle\phi\rangle = 1/G$, and request that the local scalar field ϕ also converges to $\langle\phi\rangle$ at the distant enough region ($r \rightarrow \infty$). Moreover, we adopt a selection rule, which is derived by the presence of cosmological matter in the universe. Let us consider the static spherically symmetric vacuum solution [1] in the Brans-Dicke theory (only scalar part):

$$\phi = \phi_0 \left(\frac{1 - B/r}{1 + B/r} \right)^{-C/\sigma} \quad (12)$$

where $B = (M/2C^2\phi_0)[(2\omega + 4)/(2\omega + 3)]^{1/2}$, $\sigma = [(C + 1)^2 + C(1 - \frac{1}{2}\omega C)]^{1/2}$, and C is arbitrary constant. It is obvious that this solution converges to the constant ϕ_0 in the infinite ω limit. In a case of arbitrary constant C (independent of ω), the asymptotic form of this solution becomes $\phi = \langle\phi\rangle + O(1/\sqrt{\omega})$ [3], which means that Eq.(12) does not produce the corresponding solution

of general relativity, the Schwarzschild solution. We need suppress indefiniteness of solutions with no material origin. We can not accomplish enough this work by boundary conditions. Our selection rule act to choose a proper solution (or solutions), of which asymptotic behavior is Eq.(3) with material origin, and which yields a corresponding solution of general relativity. Because the corresponding exact and global Brans-Dicke solution is originally generated by the nonvanishing energy-momentum tensor ($T_{\mu\nu} \neq 0$) by locally-distributed and cosmological matter in the universe. We can forget the effect of the presence of cosmological matter if we set the two postulates for local problems in the Brans-Dicke theory. In this standpoint, general relativity is the self-complete approximate-theory of gravitation as it needs no additional postulates.

If we take a choice $C = -1/(2 + \omega)$, the equation (12) behaves asymptotically as Eq.(3) for the large enough ω and the whole solution becomes identical with the Schwarzschild solution of the Einstein theory for $\omega \rightarrow \infty$ [1], [6]. Another choice $C = -1/2\omega$ is also available [4]. However, it is to be remarked that this ω has no meanings though the same letter ω is used. This is not the coupling parameter ω derived from the right-hand side of Eq.(2) or the second term of the right-hand side of Eq.(1). If we regard Eq.(12) as the approximate and local solution for a point-mass M at the origin and cosmological matter in the universe, we may be able to interpret this ω as the real coupling parameter derived from the right-hand side of Eq.(2) with source matter. Though this is a conjecture, let us adopt as a postulate. For local problems we should forgive only solutions of which the asymptotic behavior is $\phi = \langle \phi \rangle + O(1/\omega)$ even when $T = 0$. Thus we can restrict indefiniteness and select proper solutions with material origin. For local problems they are equivalent to each other to have material origin, to behave asymptotically as Eq.(3) for the large enough ω , and to converge to the corresponding solution of general relativity in the infinite ω limit. There exists arbitrariness of forgiven solutions, for example a choice of C , and remains controversial. They give different solutions for finite values of ω .

The contracted energy-momentum tensor T is zero for these solutions. Nevertheless the corresponding solution of general relativity is produced in the infinite ω limit. However, this should not be regarded as a counterexample to the proposition by Faraoni [4]. This proposition should be understood to stand for exact solutions with exact $T = 0$. In his standpoint, the statement $T = 0$ means that there exist completely no other matter in the universe. He treats local problems for locally-distributed matter as whole cosmological problems. To the contrary, though we use the statement $T = 0$ in these solutions, this is a local approximation for local problems and actually the complete global contracted energy-momentum tensor does not vanish

($T \neq 0$) because of the presence of other cosmological matter in the universe. If we discuss locally-distributed matter in otherwise empty space, the scalar field ϕ should not have the constant scalar field $\langle\phi\rangle$. This is the important keynote to understand the Brans-Dicke theory true. It is meaningless that we consider strictly the situation in which matter does not exist, or vacuum space in the Brans-Dicke theory.

IV Cosmological Problems

Next we consider cosmological problems to make clear further the essence of the Brans-Dicke theory. Let us discuss first the Brans-Dicke flat solution [1] for the homogeneous and isotropic universe. Assuming the initial conditions

$$\phi = a = 0; \quad t = 0, \quad (13)$$

it is given as

$$ds^2 = -dt^2 + a^2(t)[d\chi^2 + \chi^2(d\theta^2 + \sin^2\theta d\varphi^2)], \quad (14)$$

$$\phi = \phi_0(t/t_0)^r, \quad a = a_0(t/t_0)^q, \quad \rho a^3 = \rho_0 a_0^3, \quad (15)$$

with

$$r = 2/(4 + 3\omega), \quad q = (2 + 2\omega)/(4 + 3\omega), \quad (16)$$

and

$$\phi_0 = 4\pi[(4 + 3\omega)/(3 + 2\omega)c^2]\rho_0 t_0^2, \quad (17)$$

where ρ_0 is the present mass density. For the large coupling parameter ω , it is easy to observe that the scalar field ϕ of this solution behaves like Eq.(3) [6]. If the mass density ρ_0 decreases to zero, the scalar field ϕ also converges to zero, and this situation is suitable for the material origin of the scalar field. It is well-known that this solution reduces to the Einstein-de Sitter universe of general relativity in the infinite ω limit. However, what does the constant value $\langle\phi\rangle = 6\pi\rho_0 t_0^2/c^2$ mean? In the infinite ω limit, the coupling between the scalar field and matter vanishes. Why does the mass density ρ_0 appear in the constant scalar field $\langle\phi\rangle$? Which value of the density should we take? This situation is rather strange as to the material origin. After all, this constant scalar field $\langle\phi\rangle$ seems to be merely constant which has no material origin.

O'Hanlon and Tupper [8] solution for a vacuum, spatially flat Friedmann-Robertson-Walker spacetime has the asymptotic behavior $\phi = \langle\phi\rangle + O(1/\sqrt{\omega})$ [6], which means that this solution has no material origin. This solution also has the constant scalar field for $\omega \rightarrow \infty$. Nariai [9] flat solution with a

perfect fluid has the asymptotic behavior Eq.(5) for $T = 0$ (radiation), and Eq.(3) for $T \neq 0$ (matter) [3]. In both cases the scalar field converges to the constant ϕ_0 in the infinite ω limit.

The following cosmological solution [10], [11] gives both a counterexample to the theorem by Banerjee and Sen [3] and an interesting example as to the material origin of cosmological solutions. Dehnen and Obregón says that this model has no analogy in general relativity [10], but this is not adequate [12]. The Brans-Dicke theory has a particular closed solution for the homogeneous and isotropic universe with dust ($T = \rho c^2$), satisfying $a(t)\phi(t) = \text{const}$,

$$ds^2 = -dt^2 + a^2(t)[d\chi^2 + \sin^2 \chi(d\theta^2 + \sin^2 \theta d\varphi^2)], \quad (18)$$

$$\phi(t) = -[8\pi/(3 + 2\omega)c^2]\rho(t)t^2, \quad (19)$$

$$a(t) = [-2/(2 + \omega)]^{1/2} t, \quad (20)$$

$$2\pi^2 a^3(t)\rho(t) = M, \quad (21)$$

with

$$\omega < -2, \quad G(t)M/c^2 a(t) = \pi, \quad (22)$$

where the gravitational "constant" $G = (4 + 2\omega)/(3 + 2\omega)\phi$ and M is the total mass of the universe. The scalar field has obviously the asymptotic behavior $O(1/\omega)$, but does not have the constant value $\langle \phi \rangle$ in the infinite ω limit ($\omega \rightarrow -\infty$). The expansion parameter a also has the ω -dependence, which means the scalar curvature R itself has the ω -dependence.

Let us write down the nonvanishing components of the field equations (1) and (2) for the metric Eq.(18) to discuss the details of the asymptotic behavior:

$$2\dot{a}\ddot{a} + \dot{a}^2 + 1 = -\frac{8\pi}{(3 + 2\omega)c^2} \frac{a^2 \rho}{\phi} - \frac{1}{2} \omega a^2 \left(\frac{\dot{\phi}}{\phi} \right)^2 + a\dot{a} \left(\frac{\dot{\phi}}{\phi} \right), \quad (23)$$

$$\frac{3}{a^2} (\dot{a}^2 + 1) = \frac{16\pi(1 + \omega)}{(3 + 2\omega)c^2} \frac{\rho}{\phi} + \frac{\omega}{2} \left(\frac{\dot{\phi}}{\phi} \right)^2 + \frac{\ddot{\phi}}{\phi}, \quad (24)$$

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} = \frac{8\pi}{(3 + 2\omega)c^2} \rho, \quad (25)$$

where a dot denotes the derivative with respect to t . For the small enough coupling parameter ($\omega \ll -1$), we can estimate the order of each terms by means of the solution, for example,

$$\frac{8\pi}{(3 + 2\omega)c^2} \frac{a^2 \rho}{\phi} \sim O(1/\omega), \quad (26)$$

$$\frac{1}{2}\omega a^2 \left(\frac{\dot{\phi}}{\phi}\right)^2 \sim O(1), \quad (27)$$

$$a\dot{a} \left(\frac{\dot{\phi}}{\phi}\right) \sim O(1/\omega), \quad (28)$$

$$a^2 \frac{\ddot{\phi}}{\phi} \sim O(1/\omega). \quad (29)$$

Remark that the mass density ρ is given and does not depend on ω , and so M has the ω -dependence derived from a . The term Eq.(27), which is the contribution from the second term of the right-hand side of Eq.(1), remains nonvanishing even though the scalar field has the asymptotic behavior $O(1/\omega)$.

If we put $\lambda(t) \equiv -(\omega/2)(\dot{\phi}/\phi)^2$ and $\kappa(t) \equiv 8\pi/c^4\phi$, we get from Eq.(23) and Eq.(24)

$$2\dot{a}\ddot{a} + \dot{a}^2 - \lambda a^2 + 1 = O(1/\omega), \quad (30)$$

$$\frac{3}{a^2}(\dot{a}^2 + 1) + \lambda = \kappa\rho c^2 + O(1). \quad (31)$$

Thus we obtain in the infinite ω limit ($\omega \rightarrow -\infty$)

$$\lambda a^2 = 1, \quad \kappa\rho c^2 a^2 = 4. \quad (32)$$

If we regard λ as the cosmological "constant", these relations are similar to those of the static Einstein universe with negligible pressure in general relativity except the difference of the radius of the universe in $\sqrt{2}$ which is derived from the opposite sign of λ in Eq.(31). In the infinite ω limit, the expansion parameter a reduces to zero, but this is due to the initial condition $a = 0$ at $t = 0$ [12]. There exists a discrete and isolated limit at $\omega \rightarrow -\infty$. For the finite ω ($\omega < -2$) we observe

$$\lambda(t)a^2(t) = \omega/(2 + \omega), \quad \kappa(t)\rho(t)c^2 a^2(t) = 4, \quad (33)$$

and so the effective cosmological "constant" $\lambda(t)$ decreases rapidly as the universe expands.

It is remarkable that the scalar field ϕ converges to zero for both cases in which the mass density ρ goes to zero, and in which the coupling parameter ω goes continuously to the infinity ($\omega \rightarrow -\infty$). This situation is rather preferable for the material origin of the scalar field. It means that the combination of ρ/ω plays an important role there.

Now we need consider the correspondence between the Brans-Dicke theory and general relativity in combinations of $\langle\phi\rangle \neq 0$ or $\langle\phi\rangle = 0$, and $O(1/\omega)$ or $O(1/\sqrt{\omega})$. Let us put for abbreviation

$$A_{\mu\nu} \equiv \frac{\omega}{\phi^2} \left(\phi_{,\mu} \phi_{,\nu} - \frac{1}{2} g_{\mu\nu} \phi_{,\lambda} \phi^{,\lambda} \right), \quad (34)$$

$$B_{\mu\nu} \equiv \frac{1}{\phi} (\phi_{,\mu;\nu} - g_{\mu\nu} \square \phi), \quad (35)$$

$$C \equiv \frac{8\pi}{c^4 \phi}. \quad (36)$$

We can summarize orders of magnitude of each terms in ω or $\sqrt{|\omega|}$ as following:

case (i) $\phi = \langle\phi\rangle + O(1/\omega)$, $\langle\phi\rangle \neq 0$,

$$A_{\mu\nu} \sim O(1/\omega), \quad B_{\mu\nu} \sim O(1/\omega), \quad C \sim O(1), \quad (37)$$

case (ii) $\phi = \langle\phi\rangle + O(1/\sqrt{|\omega|})$, $\langle\phi\rangle \neq 0$,

$$A_{\mu\nu} \sim O(1), \quad B_{\mu\nu} \sim O(1/\sqrt{|\omega|}), \quad C \sim O(1), \quad (38)$$

case (iii) $\phi = O(1/\omega)$,

$$A_{\mu\nu} \sim O(\omega), \quad B_{\mu\nu} \sim O(1), \quad C \sim O(\omega), \quad (39)$$

case (iv) $\phi = O(1/\sqrt{|\omega|})$,

$$A_{\mu\nu} \sim O(\omega), \quad B_{\mu\nu} \sim O(1), \quad C \sim O(\sqrt{|\omega|}). \quad (40)$$

These results are derived on the assumption that the metric tensor $g_{\mu\nu}$ converges to a nonvanishing function in the infinite ω limit. If $g_{\mu\nu}$ has other ω -dependence which does not satisfy this assumption, we need another individual analysis for the specific solution, and the results seems to become different from the above. Even for local problems we can not deny this possibility. However, it is likely that $g_{\mu\nu}$ converges to a nonvanishing function in the infinite ω limit if the scalar field ϕ converges to $\langle\phi\rangle \neq 0$ for local problems. Anyhow, it is common that the Brans-Dicke solutions fail to reduce to the corresponding solutions of general relativity when $|\omega| \rightarrow \infty$.

V Discussions

Should the Brans-Dicke theory reduce to general relativity in the infinite ω limit? No longer, its statement seems to be a preconception. It is true that general relativity goes to the Newtonian theory of gravitation in the weak field approximation ($GM/Rc^2 \ll 1$). The fact that both general relativity and the Newtonian theory have the common parameter, Newton's gravitational constant G , makes it possible. However, the Brans-Dicke theory and general relativity do not have a common parameter each other. The infinite limit of the coupling parameter ω is ambiguous. After all, it is natural to realize that the Brans-Dicke theory is a different theory of gravitation from general relativity and need not reduce to it in the infinite ω limit. The differences of two theories are rather essential in the physical meaning.

There exist at least three standpoints; First, general relativity is the complete classical theory of gravitation. Second, the Brans-Dicke theory is complete and is applicable to even purely local problems. Third, the Brans-Dicke theory with some selection rules produces physically reasonable solutions.

The first standpoint is the simplest and the most real, even though we can not understand the origin of the gravitational constant G and cannot help accepting its value *a priori*. The scalar field does not exist as the gravitational field. We do not need the redundant Brans-Dicke theory. General relativity is valid completely not only for local problems but also for cosmological problems. The present experimental and observational tests strongly support this standpoint.

In the second standpoint, the opposite extreme to the first, we can freely apply the Brans-Dicke theory to all kinds of problems and formally obtain their solutions if possible. We can discuss even a vacuum space. It may be the Minkowski space. May a particle show the inertial property in this space? We may set exactly asymptotically-flatness as a boundary condition for locally-distributed matter. However, we encounter a serious difficulty owing to ambiguity with $\square\phi = 0$. This ambiguity is not avoidable as long as we consider vacuum ($T = 0$) even if we suppose specific boundary conditions for the scalar field. We obtain different solutions with the corresponding solutions of general relativity with the same energy-momentum tensor in the infinite ω limit for $T = 0$.

The third standpoint set the presence of matter forth as a premise; This is the Machian point of view. We should clearly distinguish between local and cosmological problems. Cosmology has a special situation in physics. We always live in this universe and can not test alternatives. We cannot help discussing every physical phenomena in this environment. We need adopt

selection rules for local and cosmological problems respectively. For local problems, we restrict to solutions of which *the scalar field has the asymptotic behavior* $\phi = \langle\phi\rangle + O(1/\omega)$. This selection rule is a reflection of the presence of cosmological matter in the universe and suppresses ambiguity with $\square\phi = 0$. Moreover, *the cosmological background gives the finite reasonable scalar field* $\langle\phi\rangle$, that is, the gravitational "constant" G at the present time. Thus we can handle individual local problems with asymptotically-flat boundary conditions for the metric tensor $g_{\mu\nu}$ and the boundary condition for the scalar field ϕ ($\phi \rightarrow \langle\phi\rangle$ as $r \rightarrow \infty$) in the Brans-Dicke theory without considering our environment. This is an extremely good approximation in this universe. We may discuss the space-varying G by locally-distributed matter in the "empty" space.

For cosmological problems, we adopt only a solution which behaves asymptotically like $\phi = O(\rho/\omega)$. If the inertial properties are determined completely by the presence of matter in the universe, the scalar field should not include the constant $\langle\phi\rangle$ which has no material origin. The scalar field ϕ should converge to zero when the mass density ρ decreases to zero in a proper cosmological model. The scalar field should also converge to zero when the coupling parameter $|\omega|$ diverges to infinity and the connection between the scalar field and matter vanishes. The mass density ρ and the coupling parameter ω are closely connected as the source term of Eq.(2) and so we can combine two conditions. Thus *the cosmological scalar field should converge to zero in the continuous limit* $\rho/\omega \rightarrow 0$. This situation is crucial for considering the difference between the Brans-Dicke theory and general relativity. The Minkowski space which has the constant scalar field is excluded as a proper solution. A cosmological solution which fails to reduce to the corresponding solution of general relativity is rather physically reasonable. The constant scalar $\langle\phi\rangle$ may be derived from the contribution of quantum corrections. However, this contribution should be classically renormalized to the mass density because the inertial-frame dragging is dominated completely by the distribution of the mass density itself [13].

All proper solutions for local problems in the third standpoint reduce to the corresponding solutions of general relativity in the infinite ω limit (, fixing cosmological part), so general relativity is the complete approximate-theory of gravitation in this standpoint. Rather, it may be more fundamental that we should adopt this fact as a postulate for local problems. General relativity is effective enough for local problems and for a small period for which the universe is quasi-static and the gravitational "constant" is constant enough.

It is a matter of course that experimental and observational tests should finally determine which theory (, including other extended scalar-tensor theories) is true and which standpoint is appropriate. After all, the essential

difference may appear only in cosmological problems owing to the experimentally established large coupling parameter ω . We have not known an exact solution for our universe yet.

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